

# Spherical symmetry in $f(R)$ -gravity

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Spherical symmetry in  $f(R)$  gravity is discussed in details considering also the relations with the weak field limit. Exact solutions are obtained for constant Ricci curvature scalar and for Ricci scalar depending on the radial coordinate. In particular, we discuss how to obtain results which can be consistently compared with General Relativity giving the well known post-Newtonian and post-Minkowskian limits. Furthermore, we implement a perturbation approach to obtain solutions up to the first order starting from spherically symmetric backgrounds. Exact solutions are given for several classes of  $f(R)$  theories in both  $R = \text{constant}$  and  $R = R(r)$ .

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## I. INTRODUCTION

The recent advent of new cosmological precision tests, capable of probing physics at very large redshifts, has changed the modern view of cosmos ruling out the Cosmological Standard Model based on General Relativity (GR), radiation and baryonic matter. Beside the introduction of dark matter, needed to fit the astrophysical dynamics at galactic and cluster scales (i.e. to explain clustered structures), a new ingredient is requested in order to explain the observed accelerated behavior of the Hubble flow: the dark energy.

In particular, the luminosity distance of Ia Type Supernovae [1], the Large Scale Structure [2] and the anisotropy of Cosmic Microwave Background [3] suggest that the widely accepted Cosmological Concordance Model ( $\Lambda$ CDM) is a spatially flat Universe, dominated by cold dark matter (CDM ( $\sim 0.25 \div 0.3\%$ ) which should explain the clustered structures) and dark energy ( $\Lambda \sim 0.65 \div 0.7\%$ ), in the form of an “effective” cosmological constant, giving rise to the accelerated behavior.

Although the cosmological constant [4, 5, 6] remains the most relevant candidate to interpret the accelerated behavior, several proposals have been suggested in the last few years: quintessence models, where the cosmic acceleration is generated by means of a scalar field, in a way similar to the early time inflation, acting at large scales and recent epochs [7]; models based on exotic fluids like the Chaplygin-gas [8], or non-perfect fluids [9]; phantom fields, based on scalar fields with anomalous signature in the kinetic term [10], higher dimensional scenarios (braneworld) [11]. Actually, all of these models, are based on the peculiar characteristic of introducing new sources into the cosmological dynamics, while it would be preferable to develop scenarios consistent with observations without invoking further parameters or components non-testable (up to now) at a fundamental level.

The resort to modified gravity theories, which extend in some way the GR, allows to pursue this different approach (no further unknown sources) giving rise to suitable cosmological models where a late time accelerated expansion naturally arises.

The idea that the Einstein gravity should be extended or corrected at large scales (infrared limit) or at high energies (ultraviolet limit) is suggested by several theoretical and observational issues. Quantum field theories in curved spacetimes, as well as the low energy limit of string theory, both imply semi-classical effective Lagrangians containing higher-order curvature invariants or scalar-tensor terms. In addition, GR has been tested only at solar system scales while it shows several shortcomings if checked at higher energies or larger scales.

Of course modifying the gravitational action asks for several fundamental challenges. These models can exhibit instabilities [13] or ghost-like behaviors [14], while, on the other side, they should be matched with the low energy limit observations and experiments (solar system tests, PPN limit). Despite of all these issues, in the last years,

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several interesting results have been achieved in the framework of the so called  $f(R)$ - gravity at cosmological, galactic and solar system scales.

For example, models based on generic functions of the Ricci scalar  $R$  show cosmological solution with late time accelerating dynamics [16], in addition, it has been shown that some of them could agree with CMBR observational prescriptions [17], nevertheless this matter is still argument of debate [18]. For a review of the models and their cosmological applications see, e.g., [19, 20].

Moreover, considering  $f(R)$ -gravity in the low energy limit, it is possible to obtain corrected gravitational potentials capable of explaining the flat rotation curves of spiral galaxies without considering huge amounts of dark matter [21, 22, 23, 24] and, furthermore, this seems the only self-consistent way to reproduce the universal rotation curve of spiral galaxies [25]. On the other hand, several anomalies in solar system experiments could be framed and addressed in this picture [26].

In the last few years, several authors have dealt with this matter considering the Parameterized Post Newtonian (PPN) limit [27]. On the other hand, the investigation of spherically symmetric solutions for such kind of models has been developed in several papers [28, 30, 31, 32]. Such an analysis deserves particular attention since it can allow to draw interesting conclusions on the effective modification of the gravitational potential induced by higher order gravity at low energies and, in addition, it could shed new light on the PPN limit of such theories. However, in several recent papers, it has been shown that several classes of  $f(R)$  models fairly evade the Solar System constraints and agree with the limits of PPN-parameters, in particular with  $\gamma \sim 1$  probed by experiments [35, 36, 37, 38, 39]. This is not the argument of the present paper, however it is important to stress that the spherical symmetry and the weak field limit have to be carefully considered in order to find out physically viable models.

In this paper, we are going to analyze, in a general way and without specifying *a priori* the form of the Lagrangian, the relation between the spherical symmetry and the weak field limit of  $f(R)$  gravity. Our aim is to develop a systematic approach considering the theoretical prescriptions to obtain a correct weak field limit in order to point out the analogies and the differences with respect to GR. A fundamental issue is to recover the asymptotically flat solution in absence of gravity and the well-known results related to the specific case  $f(R) = R$ , i.e. GR. Only in this situation a correct comparison between GR and any extended gravity theory is possible from an experimental and a theoretical viewpoint. For example, as already shown in [34] and discussed in [40], the Birkhoff theorem is not a general result for fourth-order models also if it holds for several interesting classes of these theories as discussed, for example, in [32, 41, 42].

The layout of the paper is the following. Sec.II is devoted to some general remarks on spherical symmetry in  $f(R)$  gravity. In particular, we derive the field equations in such a symmetry. In Sec.III, the expression of the Ricci scalar and the general form of metric components are derived in spherical symmetry discussing how recovering the correct Minkowski flat limit. Sec.IV is devoted to the discussion of spherically symmetric background solutions with constant scalar curvature considering, in particular Schwarzschild-like and Schwarzschild-de Sitter-like solutions with constant curvature. Some remarkable  $f(R)$  models are worked out as example. In Sec. V, we discuss the cases in which the spherical symmetry is present also for the Ricci scalar depending on the radial coordinate  $r$ . This is an interesting situation, not present in GR. In fact, as it is well known, in the Einstein theory, the Birkhoff theorem states that a spherically symmetric solution is always stationary and static [44] and the Ricci scalar is constant. In  $f(R)$  the situation is more general and then the Ricci scalar, in principle, can evolve with radial and time coordinates. Sec.VI is devoted to the study of a perturbation approach starting from a spherically symmetric background considering the general case in which the Ricci scalar is  $R = R(r)$ . The motivation is due to the fact that, in GR, the Schwarzschild solution and the weak field limit coincide under suitable conditions. This could be a good test bed to construct a well-posed weak field limit for any extended theory of gravity. Finally, we work out some  $f(R)$ -models finding spherically-symmetric solutions by the perturbation approach. Conclusions are drawn in Sec.VII.

## II. SPHERICAL SYMMETRY

Let us consider an analytic function  $f(R)$  of the Ricci scalar  $R$ . The variational principle for this action is:

$$\delta \int d^4x \sqrt{-g} \left[ f(R) + \mathcal{X} \mathcal{L}_m \right] = 0 \quad (1)$$

where  $\mathcal{X} = \frac{16\pi G}{c^4}$ ,  $\mathcal{L}_m$  is the standard matter Lagrangian and  $g$  is the determinant of the metric. By varying with

respect to the metric, we obtain the field equations <sup>1</sup>

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\square f'(R) = \frac{\mathcal{X}}{2}T_{\mu\nu}, \quad (2)$$

where the trace equation read

$$3\square f'(R) + f'(R)R - 2f(R) = \frac{\mathcal{X}}{2}T, \quad (3)$$

and  $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$  is the energy momentum tensor of standard matter,  $T = T^\sigma{}_\sigma$  is the related trace and  $f'(R) = \frac{df(R)}{dR}$ . Eqs.(2) can be rewritten in an Einstein-like form recasting the higher than second order contributions under the form of an effective stress-energy tensor of geometrical origin:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^{(curv)} + T_{\mu\nu}^{(m)}. \quad (4)$$

The effective curvature stress-energy tensor is [15]:

$$T_{\mu\nu}^{(curv)} = \frac{1}{f'(R)} \{g_{\mu\nu} [f(R) - Rf'(R)] + f'(R)^{;\rho\sigma} (g_{\mu\rho}g_{\nu\sigma} - g_{\rho\sigma}g_{\mu\nu})\}. \quad (5)$$

The matter term enters Eqs.(4) through the modified stress-energy tensor:

$$T_{\mu\nu}^{(m)} = \frac{\mathcal{X}T_{\mu\nu}}{2f'(R)}, \quad (6)$$

where there is a non-minimal coupling between matter and geometric degrees of freedom. For the purposes of this paper, we will refer to the field equations in the form (2) which reveal to be more useful for the following considerations.

The most general spherically symmetric metric can be written as follows:

$$ds^2 = m_1(t', r')dt'^2 + m_2(t', r')dr'^2 + m_3(t', r')dt'dr' + m_4(t', r')d\Omega, \quad (7)$$

where  $m_i$  are functions of the radius  $r'$  and of the time  $t'$  (see also [32]). In GR, due to the freedom in the frame choice, we can consider a coordinate transformation  $t = U_1(t', r')$ ,  $r = U_2(t', r')$  which maps the metric (7) in a new one where the off-diagonal term  $m_3(t', r')dt'dr'$  vanishes and  $m_4(t', r') = -r^2$ , that is<sup>2</sup>:

$$ds^2 = A(t, r)dt^2 - B(t, r)dr^2 - r^2d\Omega. \quad (8)$$

This expression can be considered without loss of generality as the most general definition of a spherically symmetric metric compatible with a pseudo-Riemannian manifold without torsion. Actually, by inserting this metric into the field equations (2) and (3), one obtains:

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \frac{\mathcal{X}}{2}T_{\mu\nu}, \quad (9)$$

$$g^{\sigma\tau}H_{\sigma\tau} = f'(R)R - 2f(R) + \mathcal{H} = \frac{\mathcal{X}}{2}T, \quad (10)$$

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<sup>1</sup> It is possible to take into account also the Palatini approach in which the metric  $g$  and the connection  $\Gamma$  are considered independent variables (see for example [45]). Here we will consider the Levi-Civita connection and will use the metric approach. See [19, 46] for a detailed comparison between the two pictures.

<sup>2</sup> This condition allows to obtain the standard definition of the circumference with radius  $r$ .

where the two quantities  $\mathcal{H}_{\mu\nu}$  and  $\mathcal{H}$  read:

$$\mathcal{H}_{\mu\nu} = -f''(R) \left\{ R_{,\mu\nu} - \Gamma_{\mu\nu}^t R_{,t} - \Gamma_{\mu\nu}^r R_{,r} - g_{\mu\nu} \left[ \left( g^{tt}_{,t} + g^{tt} \ln \sqrt{-g}_{,t} \right) R_{,t} + \left( g^{rr}_{,r} + g^{rr} \ln \sqrt{-g}_{,r} \right) R_{,r} + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] \right\} - f'''(R) \left[ R_{,\mu} R_{,\nu} - g_{\mu\nu} \left( g^{tt} R_{,t}^2 + g^{rr} R_{,r}^2 \right) \right] \quad (11)$$

$$\mathcal{H} = g^{\sigma\tau} \mathcal{H}_{\sigma\tau} = 3f''(R) \left[ \left( g^{tt}_{,t} + g^{tt} \ln \sqrt{-g}_{,t} \right) R_{,t} + \left( g^{rr}_{,r} + g^{rr} \ln \sqrt{-g}_{,r} \right) R_{,r} + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] + 3f'''(R) \left[ g^{tt} R_{,t}^2 + g^{rr} R_{,r}^2 \right], \quad (12)$$

where  $\Gamma_{\mu\nu}^\alpha$  are the standard Christoffel's symbols related to the metric  $g_{\mu\nu}$ . These equations show the interesting feature that the derivatives with respect to  $R$  of  $f(R)$  are very well distinct with respect to the time and spatial derivatives of  $R$ . This peculiarity will allow to better understand the dynamical behavior of solutions, as we shall see below.

### III. THE RICCI CURVATURE SCALAR IN SPHERICAL SYMMETRY

As standard, the Ricci scalar can be written as

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \left[ \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\alpha\beta}^\beta \Gamma_{\mu\nu}^\alpha - \Gamma_{\beta\mu}^\alpha \Gamma_{\nu\alpha}^\beta \right], \quad (13)$$

where  $\Gamma_{\mu\nu}^\alpha$  is the Christoffel symbols defined as

$$\Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}). \quad (14)$$

Imposing the spherical symmetry (8), the Ricci scalar in terms of the gravitational potentials reads:

$$R(t, r) = \frac{B \left[ \dot{A}\dot{B} - A'^2 \right] r^2 + A \left[ r(\dot{B}^2 - A'B') + 2B(2A' + rA'' - r\ddot{B}) \right] - 4A^2 \left[ B^2 - B + rB' \right]}{2r^2 A^2 B^2} \quad (15)$$

where, for sake of brevity, we have discarded the explicit dependence in  $A(t, r)$  and  $B(t, r)$  and the prime indicates the derivative with respect to  $r$  while the dot with respect to  $t$ . If the metric (8) is time-independent, i.e.  $A(t, r) = a(r)$ ,  $B(t, r) = b(r)$ , Eq.(15) assumes the simpler form:

$$R(r) = \frac{a(r) \left[ 2b(r) \left( 2a'(r) + ra''(r) \right) - ra'(r)b'(r) \right] - b(r)a'(r)^2 r^2 - 4a(r)^2 \left( b(r)^2 - b(r) + rb'(r) \right)}{2r^2 a(r)^2 b(r)^2} \quad (16)$$

where the radial dependence of the gravitational potentials is now explicitly shown. This expression can be seen as a constraint for the functions  $a(r)$  and  $b(r)$  once a specific form of Ricci scalar is given. In particular, it reduces to a Bernoulli equation of index two, that is

$$b'(r) + h(r)b(r) + l(r)b(r)^2 = 0,$$

with respect to the metric potential  $b(r)$ :

$$b'(r) + \left\{ \frac{r^2 a'(r)^2 - 4a(r)^2 - 2ra(r)[2a(r)' + ra(r)']}{ra(r)[4a(r) + ra'(r)]} \right\} b(r) + \left\{ \frac{2a(r)}{r} \left[ \frac{2 + r^2 R(r)}{4a(r) + ra'(r)} \right] \right\} b(r)^2 = 0. \quad (17)$$

A general solution of (17) is:

$$b(r) = \frac{\exp[-\int dr h(r)]}{K + \int dr l(r) \exp[-\int dr h(r)]}, \quad (18)$$

where  $K$  is an integration constant while  $h(r)$  and  $l(r)$  are the two functions which, respectively, define the coefficients of the quadratic and the linear term with respect to  $b(r)$ , as in the standard definition of the Bernoulli equation [47]. Looking at the equation, we can notice that it is possible to have  $l(r) = 0$  which implies to find out solutions with a Ricci scalar scaling as  $-\frac{2}{r^2}$  in term of the radial coordinate. On the other side, it is not possible to have  $h(r) = 0$  since otherwise we will get imaginary solutions. A particular consideration deserves the limit  $r \rightarrow \infty$ . In order to achieve a gravitational potential  $b(r)$  with the correct Minkowski limit, both  $h(r)$  and  $l(r)$  have to go to zero provided that the quantity  $r^2 R(r)$  turns out to be constant: this result implies  $b'(r) = 0$ , and, finally, also the metric potential  $b(r)$  has a correct Minkowski limit.

In general, if we ask for the asymptotic flatness of the metric as a feature of the theory, the Ricci scalar has to evolve at infinity as  $r^{-n}$  with  $n \geq 2$ . Formally it has to be:

$$\lim_{r \rightarrow \infty} r^2 R(r) = r^{-n} \quad (19)$$

with  $n \in \mathbb{N}$ . Any other behavior of the Ricci scalar could affect the requirement of the correct asymptotic flatness. This result can be easily deduced from Eq.(17). In fact, let us consider the simplest spherically symmetric case:

$$ds^2 = a(r)dt^2 - \frac{dr^2}{a(r)} - r^2 d\Omega. \quad (20)$$

The Bernoulli Eq.(17) is easy to integrate and the most general metric potential  $a(r)$ , compatible with the Ricci scalar constraint (16), is:

$$a(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{1}{r^2} \int \left[ \int r^2 R(r) dr \right] dr \quad (21)$$

where  $k_1$  and  $k_2$  are integration constants. Actually one gets the standard result  $a(r) = 1$  (Minkowski) for  $r \rightarrow \infty$  only if the condition (19) is satisfied, otherwise we get a diverging gravitational potential.

#### IV. SOLUTIONS WITH CONSTANT CURVATURE SCALAR

The case of constant curvature is equivalent to GR with a cosmological constant and the solution is time independent. This result is well known (see, for example, [48]) but we report, for the sake of completeness, some considerations related with it in order to deal with more general case where a radial dependence is supposed. Let us assume a scalar curvature constant ( $R = R_0$ ). The field Eqs.(9) and (10), being  $\mathcal{H}_{\mu\nu} = 0$ , reduce to:

$$f'_0 R_{\mu\nu} - \frac{1}{2} f_0 g_{\mu\nu} = \frac{\mathcal{X}}{2} T_{\mu\nu}, \quad (22)$$

$$f'_0 R_0 - 2f_0 = \frac{\mathcal{X}}{2} T. \quad (23)$$

where  $f(R_0) = f_0$ ,  $f'(R_0) = f'_0$ . Such equations can be arranged as:

$$R_{\mu\nu} + \lambda g_{\mu\nu} = q \frac{\mathcal{X}}{2} T_{\mu\nu} \quad (24)$$

$$R_0 = q \frac{\mathcal{X}}{2} T - 4\lambda \quad (25)$$

where  $\lambda = -\frac{f_0}{2f'_0}$  and  $q^{-1} = f'_0$ . As standard, the stress-energy tensor of perfect-fluid matter is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}; \quad (26)$$

where  $\rho$  is the energy density,  $p$  is the pressure and  $u^\mu = dx^\mu/ds$  is the 4-velocity. A general solution of the above set of equations is achieved for  $p = -\rho$  and reads:

$$ds^2 = \left(1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - 2\lambda}{6}r^2\right)dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - 2\lambda}{6}r^2} - r^2 d\Omega. \quad (27)$$

This result means that any  $f(R)$ -theory in the case of constant curvature scalar ( $R = R_0$ ) exhibits solutions with cosmological constant-like behavior (de Sitter). This is one of the reason why the dark energy issue can be addressed using these theory [15, 16]. This fact is well known using the FRW metric [48]. As another remark we have to say that this solution cannot describe stellar structures since this kind of equation of state does not work for stars so it could be interesting only in cosmological contexts.

If  $f(R)$  is analytic, it is:

$$f(R) = \Lambda + \Psi_0 R + \Psi(R). \quad (28)$$

where  $\Psi_0$  is a coupling constant,  $\Lambda$  plays the role of the cosmological constant and  $\Psi(R)$  is a generic analytic function of  $R$  satisfying the condition

$$\lim_{R \rightarrow 0} R^{-2}\Psi(R) = \Psi_1 \quad (29)$$

where  $\Psi_1$  is a constant. If we neglect the cosmological constant  $\Lambda$  and  $\Psi_0$  is set to zero, we obtain a new class of theories which, in the limit  $R \rightarrow 0$ , does not reproduce GR (from Eq.(29), we have  $\lim_{R \rightarrow 0} f(R) \sim R^2$ ). In such a case analyzing the whole set of Eqs.(22) and (23), one can observe that both zero and constant  $\neq 0$  curvature solutions are possible. In particular, if  $R = R_0 = 0$  field equations are solved for every form of gravitational potentials entering the spherically symmetric background, provided that the Bernoulli equation (17), relating these functions, is fulfilled for the particular case  $R(r) = 0$ . The solutions are thus defined by the relation

$$b(r) = \frac{\exp[-\int dr h(r)]}{K + 4 \int \frac{dr a(r) \exp[-\int dr h(r)]}{r[a(r) + ra'(r)]}}, \quad (30)$$

being  $B(t, r) = b(r)$  from Eq.(8). In Table I, we give some examples of  $f(R)$ -theories admitting solutions with constant  $\neq 0$  or null scalar curvature. Each model admits Schwarzschild, Schwarzschild-de Sitter, and the class of solutions given by (30).

## V. SOLUTIONS WITH CURVATURE SCALAR FUNCTION OF $r$

Up to now we have discussed the behavior of  $f(R)$  gravity seeking for spherically symmetric solutions with constant scalar curvature. This situation is well known in GR and give rise to the Schwarzschild solution ( $R = 0$ ) and the Schwarzschild-de Sitter solution ( $R = R_0 \neq 0$ ). The problem can be generalized in  $f(R)$  gravity considering the Ricci scalar as an arbitrary function of the radial coordinate  $r$ .

This approach is interesting since, in general, Higher Order Gravity theories are supposed to admit such kind of solutions and several examples have been found in literature [14, 21, 30, 32]. Here we want to face the problem from general point of view.

If we choose the Ricci scalar  $R$  as a generic function of the radial coordinate ( $R = R(r)$ ), it is possible to show that also in this case the solution of the field Eqs.(9) and (10) is time independent (if  $T_{\mu\nu} = 0$ ). In other words, the Birkhoff theorem has to hold. The crucial point of the approach is to study the off-diagonal  $\{t, r\}$ -component of (9) as well as in the case of GR. This equation, for a generic  $f(R)$  reads:

$$\frac{d}{dr} \left( r^2 f'(R) \right) \dot{B}(t, r) = 0, \quad (31)$$

$f(R)$ - theory:		Field equations:
$R$	$\longrightarrow$	$R_{\mu\nu} = 0$ , with $R = 0$
$\xi_1 R + \xi_2 R^n$	$\longrightarrow$	$\begin{cases} R_{\mu\nu} = 0 & \text{with } R = 0, \xi_1 \neq 0 \\ R_{\mu\nu} + \lambda g_{\mu\nu} = 0 & \text{with } R = \left[ \frac{\xi_1}{(n-2)\xi_2} \right]^{\frac{1}{n-1}}, \xi_1 \neq 0, n \neq 2 \\ 0 = 0 & \text{with } R = 0, \xi_1 = 0 \\ R_{\mu\nu} + \lambda g_{\mu\nu} = 0 & \text{with } R = R_0, \xi_1 = 0, n = 2 \end{cases}$
$\xi_1 R + \xi_2 R^{-m}$	$\longrightarrow$	$R_{\mu\nu} + \lambda g_{\mu\nu} = 0$ with $R = \left[ -\frac{(m+2)\xi_2}{\xi_1} \right]^{\frac{1}{m+1}}$
$\xi_1 R + \xi_2 R^n + \xi_3 R^{-m}$	$\longrightarrow$	$R_{\mu\nu} + \lambda g_{\mu\nu} = 0$ , with $R = R_0$ so that $\xi_1 R_0^{m+1} + (2-n)\xi_2 R_0^{n+m} + (m+2)\xi_3 = 0$
$\frac{R}{\xi_1 + R}$	$\longrightarrow$	$\begin{cases} R_{\mu\nu} = 0 & \text{with } R = 0 \\ R_{\mu\nu} + \lambda g_{\mu\nu} = 0 & \text{with } R = -\frac{\xi_1}{2} \end{cases}$
$\frac{1}{\xi_1 + R}$	$\longrightarrow$	$R_{\mu\nu} + \lambda g_{\mu\nu} = 0$ , with $R = -\frac{2\xi_1}{3}$

TABLE I: Examples of  $f(R)$ -models admitting constant and zero scalar curvature solutions. In the right hand side, the field equations are given for each model. The power  $n, m$  are natural numbers while  $\xi_i$  are generic real constants.

and two possibilities are in order. Firstly, we can choose  $\dot{B}(t, r) \neq 0$ . This choice implies that  $f'(R) \sim \frac{1}{r^2}$ . If this is the case, the remaining field equation turn out to be not fulfilled and it can be easily recognized that the dynamical system encounters a mathematical incompatibility.

The only possible solution is given by  $\dot{B}(t, r) = 0$  and then the gravitational potential has to be  $B(t, r) = b(r)$ . Considering also the  $\{\theta, \theta\}$ -equation one can determine that the gravitational potential  $A(t, r)$  can be factorized with respect to the time, so that we get solutions which can be recast in the stationary spherically symmetric form after a suitable coordinate transformation.

As a matter of fact, even the more general radial dependent case admit time-independent solutions. From the trace equation and the  $\{\theta, \theta\}$ -component, we deduce a relation which links  $A(t, r) = a(r)$  and  $B(t, r) = b(r)$ :

$$a(r) = \frac{b(r) e^{\frac{2}{3} \int \frac{(Rf' - 2f)b(r)}{R'f''} dr}}{r^4 R'^2 f''^2}, \quad (32)$$

(with  $f'' \neq 0$ ) and one which relates  $b(r)$  and  $f(R)$  (see also [30] for a similar result):

$$b(r) = \frac{6[f'(rR'f'')' - rR'^2 f''^2]}{rf(rR'f'' - 4f') + 2f'(rR(f' - rR'f'') - 3R'f'')}. \quad (33)$$

As above, three further equations has to be satisfied to completely solve the system (respectively the  $\{t, t\}$  and  $\{r, r\}$  components of the field equations and the Ricci scalar constraint) while the only unknown functions are  $f(R)$  and the Ricci scalar  $R(r)$ .

If we now consider a fourth order model of the form  $f(R) = R + \Phi(R)$ , with  $\Phi(R) \ll R$  we are capable of satisfying the whole set of equations up to third order in  $\Phi$ . In particular, we can solve the whole set of equations: the relations (32) and (33) will give the general solution depending only on the forms of  $\Phi(R)$  and  $R = R(r)$ , that is:

$$a(r) = \frac{b(r)e^{-\frac{2}{3} \int \frac{[R + (2\Phi - R\Phi')]b(r)}{R'\Phi''} dr}}{r^4 R'^2 \Phi'^2} \quad (34)$$

$$b(r) = -\frac{3(rR'\Phi'')_{,r}}{rR}. \quad (35)$$

This solution is one of the main results of this paper. In fact, once the radial dependence of the scalar curvature is obtained, Eqs.(34) allow to write down the solution of the field equations and the gravitational potential, related to the function  $a(r)$ , can be deduced. Furthermore one can check the physical relevance of such a potential by means of astrophysical data, see for example the analysis in [31]. As a final remark, we have to stress that the solution (34) has been achieved here in the vacuum case but, in general,  $R(r)$  should be obtained from the source  $T$  using the trace Eq.(3). This task is simpler in Palatini approach than in the metric approach since, in the former case, the box operator is not present in the trace equation. For a detailed discussion, see [28, 29].

## VI. PERTURBING THE SPHERICALLY SYMMETRIC SOLUTIONS

The search for solutions in  $f(R)$ -gravity, in the case of Ricci scalar dependent on the radial coordinate, can be faced by means of a perturbation approach. There are several perturbation techniques by which higher order gravity can be investigated in the weak field limit. A general approach is starting from analytical  $f(R)$  theories assuming that the background model slightly deviates from the Einstein GR (this means to consider  $f(R) = R + \Phi(R)$  where  $\Phi(R) \ll R$  as above). Another approach can be developed starting from the background metric considered as the 0th-order solution. Both these approaches assume the weak field limit of a given higher order gravity theory as a correction to the Einstein GR, supposing that zero order approximation should yield the standard lore.

Both these methods can provide interesting results on the astrophysical scales where spherically symmetric solutions characterized by small values of the scalar curvature, can be taken into account.

In the following, we will consider the first approach assuming that the background metric matches, at zero order, the GR solutions.

In general, searching for solutions by a perturbation technique means to perturb the metric  $g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)}$ . This implies that the field equations (9) and (10) split, up to first order, in two levels. Besides, a perturbation on the metric acts also on the Ricci scalar  $R$  (see the relation (13)) and then we can Taylor expand the analytic  $f(R)$  about the background value of  $R$ , i.e.:

$$f(R) = \sum_n \frac{f^{n(0)}}{n!} \left[ R - R^{(0)} \right]^n = \sum_n \frac{f^{n(0)}}{n!} R^{(1)n}. \quad (36)$$

However the above condition  $\Phi(R) \ll R$  has to imply the validity of the linear approximation  $f''(R^{(0)})/f'(R^{(0)})R^{(1)} \ll 1$ . This is demonstrated by assuming  $f'(R) = 1 + \Phi'(R)$  and  $f''(R) = \Phi''(R)$ . Immediately we obtain that the condition is fulfilled for

$$\frac{\Phi''(R^{(0)})R^{(1)}}{1 + \Phi'(R^{(0)})} \ll 1. \quad (37)$$

For example, given a Lagrangian of the form  $f(R) = R + \frac{\mu}{R}$ , it means

$$\frac{2\mu R^{(1)}}{R^{(0)}(R^{(0)2} - \mu)} \ll 1, \quad (38)$$

while, for  $f(R) = R + \alpha R^2$ , it is

$$\frac{2\alpha R^{(1)}}{1 + 2\alpha R^{(0)}} \ll 1. \quad (39)$$

This means that the validity of the approximation strictly depends on the form of the models and the value of the parameters, in the previous case  $\mu$  and  $\alpha$ . For the considerations below, we will assume that it holds. A detailed discussion for the Palatini formalism is in [28, 29].



The zero order field equations read :

$$f'^{(0)} R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} f^{(0)} + \mathcal{H}_{\mu\nu}^{(0)} = \frac{\mathcal{X}}{2} T_{\mu\nu}^{(0)} \quad (40)$$

where

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(0)} = & -f''^{(0)} \left\{ R_{,\mu\nu}^{(0)} - \Gamma^{(0)\rho}_{\mu\nu} R_{,\rho}^{(0)} - g_{\mu\nu}^{(0)} \left( g^{(0)\rho\sigma}_{,\rho} R_{,\sigma}^{(0)} + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \ln \sqrt{-g}_{,\rho} R_{,\sigma}^{(0)} \right) \right\} + \\ & -f'''^{(0)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \right\}. \end{aligned} \quad (41)$$

At first order one has :

$$f'^{(0)} \left\{ R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} \right\} + f''^{(0)} R_{\mu\nu}^{(1)} - \frac{1}{2} f^{(0)} g_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(1)} = \frac{\mathcal{X}}{2} T_{\mu\nu}^{(1)} \quad (42)$$

with

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(1)} = & -f''^{(0)} \left\{ R_{,\mu\nu}^{(1)} - \Gamma^{(0)\rho}_{\mu\nu} R_{,\rho}^{(1)} - \Gamma^{(1)\rho}_{\mu\nu} R_{,\rho}^{(0)} - g_{\mu\nu}^{(0)} \left[ g^{(0)\rho\sigma}_{,\rho} R_{,\sigma}^{(1)} + g^{(1)\rho\sigma}_{,\rho} R_{,\sigma}^{(0)} + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(1)} + \right. \right. \\ & + g^{(1)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \left( \ln \sqrt{-g}_{,\rho}^{(0)} R_{,\sigma}^{(1)} + \ln \sqrt{-g}_{,\rho}^{(1)} R_{,\sigma}^{(0)} \right) + g^{(1)\rho\sigma} \ln \sqrt{-g}_{,\rho}^{(0)} R_{,\sigma}^{(0)} \left. \right] - g_{\mu\nu}^{(1)} \left( g^{(0)\rho\sigma}_{,\rho} R_{,\sigma}^{(0)} + \right. \\ & + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \ln \sqrt{-g}_{,\rho}^{(0)} R_{,\sigma}^{(0)} \left. \right) \left. \right\} - f'''^{(0)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(1)} + R_{,\mu}^{(1)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} \left( R_{,\rho}^{(0)} R_{,\sigma}^{(1)} + R_{,\rho}^{(1)} R_{,\sigma}^{(0)} \right) + \right. \\ & - g_{\mu\nu}^{(0)} g^{(1)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} - g_{\mu\nu}^{(1)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \left. \right\} - f'''^{(0)} R^{(1)} \left\{ R_{,\mu\nu}^{(0)} - \Gamma^{(0)\rho}_{\mu\nu} R_{,\rho}^{(0)} - g_{\mu\nu}^{(0)} \left( g^{(0)\rho\sigma}_{,\rho} R_{,\sigma}^{(0)} + \right. \right. \\ & + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \ln \sqrt{-g}_{,\rho}^{(0)} R_{,\sigma}^{(0)} \left. \right) \left. \right\} - f^{IV(0)} R^{(1)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \right\}. \end{aligned} \quad (43)$$

A part the analyticity, no hypothesis has been invoked on the form of  $f(R)$ . As a matter of fact,  $f(R)$  can be completely general. At this level, to solve the problem, it is required the zero order solution of Eqs.(40) which, in general, could be a GR solution. This problem can be overcome assuming the same order of perturbation on the  $f(R)$ , that is:

$$f = R + \Phi(R), \quad (44)$$

where  $\Phi(R)$  is a generic function of the Ricci scalar fulfilling the prescription  $\Phi \ll R$ . Then we have

$$f = R^{(0)} + R^{(1)} + \Phi^{(0)}, \quad f' = 1 + \Phi'^{(0)}, \quad f'' = \Phi''^{(0)}, \quad f''' = \Phi'''^{(0)}, \quad (45)$$

and Eqs.(40) reduce to the form

$$R_{\mu\nu}^{(0)} - \frac{1}{2} R^{(0)} g_{\mu\nu}^{(0)} = \frac{\mathcal{X}}{2} T_{\mu\nu}^{(0)}. \quad (46)$$

On the other hand, Eqs.(42) reduce to

$$R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} - \frac{1}{2} g_{\mu\nu}^{(1)} R^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} \Phi^{(0)} + \Phi'^{(0)} R_{\mu\nu}^{(0)} + \mathcal{H}_{\mu\nu}^{(1)} = \frac{\mathcal{X}}{2} T_{\mu\nu}^{(1)} \quad (47)$$

where

$$\mathcal{H}_{\mu\nu}^{(1)} = -\Phi'''^{(0)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)rr} R_{,r}^{(0)} R_{,r}^{(0)} \right\} - \Phi''^{(0)} \left\{ R_{,\mu\nu}^{(0)} - \Gamma_{\mu\nu}^{(0)r} R_{,r}^{(0)} - g_{\mu\nu}^{(0)} \left( g^{(0)rr}{}_{,r} R_{,r}^{(0)} + \right. \right. \\ \left. \left. + g^{(0)rr} R_{,rr}^{(0)} + g^{(0)rr} \ln \sqrt{-g_{,r}^{(0)}} R_{,r}^{(0)} \right) \right\}. \quad (48)$$

The new system of field equations is evidently simpler than the starting one and once the zero order solution is obtained, the solutions at the first order correction can be easily achieved. In Table II, a list of solutions, obtained with this perturbation method, is given considering different classes of  $f(R)$  models.

Some remarks on these solutions are in order at this point. In the case of  $f(R)$  models which are evidently corrections to the Hilbert-Einstein Lagrangian as  $\Lambda + R + \epsilon R \ln R$  and  $R + \epsilon R^n$ , with  $\epsilon \ll 1$ , one obtains exact solutions for the gravitational potentials  $a(r)$  and  $b(r)$  related by  $a(r) = b(r)^{-1}$ . The first order expansion is straightforward as in GR. If the functions  $a(r)$  and  $b(r)$  are not related, for  $f(R) = \Lambda + R + \epsilon R \ln R$ , the first order system is directly solved without any prescription on the perturbation functions  $x(r)$  and  $y(r)$ . This is not the case for  $f(R) = R + \epsilon R^n$  since, for this model, one can obtain an explicit constraint on the perturbation function implying the possibility to deduce the form of the gravitational potential  $\phi(r)$  from  $a(r) = 1 + \frac{2\phi(r)}{c^2}$ . In such a case, no corrections are found with respect to the standard Newtonian potential. The theories  $f(R) = R^n$  and  $f(R) = \frac{R}{(\xi + R)}$  show similar behaviors. The case  $f(R) = R^2$  is peculiar and it has to be dealt independently.

$f(R)$ - theory:	$\Lambda + R + \epsilon R \ln R$
spherical potentials:	$a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} - \frac{\Lambda r^2}{6} + \delta x(r)$
solutions:	$x(r) = \frac{k_2}{r} + \frac{\epsilon \Lambda [\ln(-2\Lambda) - 1] r^2}{6\delta}$
first order metric:	$a(r) = 1 - \frac{\Lambda r^2}{6} + \delta x(r), \quad b(r) = \frac{1}{1 - \frac{\Lambda r^2}{6}} + \delta y(r)$
solutions:	$\begin{cases} x(r) = (\Lambda r^2 - 6) \left\{ k_1 + \int dr \frac{4\delta(2\Lambda^2 r^4 - 15\Lambda r^2 + 18)y(r) + r\{36r\epsilon\Lambda[\log(-2\Lambda) - 1] + \delta(\Lambda r^2 - 6)^2 y'(r)\}}{36r\delta(\Lambda r^2 - 6)} \right\} \\ y(r) = \frac{k_2\delta - 6r^3\epsilon\Lambda[\ln(-2\Lambda) - 1]}{r\delta(r^2\Lambda - 6)^2} \end{cases}$
$f(R)$ - theory:	$R + \epsilon R^n$
spherical potentials:	$a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} + \delta x(r)$
solutions:	$x(r) = \frac{k_2}{r}$
first order metric:	$a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$
solutions:	$x(r) = k_1 + k_2 r, \quad y(r) = k_3$
$f(R)$ - theory:	$R^n$
spherical potentials:	$a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} + \frac{R_0 r^2}{12} + \delta x(r)$
solutions:	$\begin{cases} n = 2, & R_0 \neq 0 \text{ and } x(r) = \frac{3k_2 - k_3}{3r} + \frac{k_3 r^2}{12} + \frac{k_4}{r} \int dr r^2 \left\{ \int dr \frac{\exp\left[\frac{R_0 r_0^2 \ln(r - r_0)}{8 + 3R_0 r_0^2}\right]}{r^5} \right\} \\ & \text{with } r_0 \text{ satisfying the condition } 6k_1 + 8r_0 + R_0 r_0^3 = 0 \\ n \geq 2, & \text{System solved only whit } R_0 = 0 \text{ and no prescriptions on } x(r) \end{cases}$
first order metric:	$a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$
solutions:	$\begin{cases} n = 2 & y(r) = -\frac{R_0 r^3}{6} - \frac{x(r)}{2} + \frac{1}{2} r x'(r) + k_1, \quad R(r) = \delta R_0 \\ n \neq 2 & y(r) = -\frac{1}{2} \int dr r^2 R(r) - \frac{x(r)}{2} + \frac{1}{2} r x'(r) + k_1 \text{ with } R(r) \text{ whatever} \end{cases}$
first order metric:	$a(r) = 1 - \frac{r_g}{r} + \delta x(r), \quad b(r) = \frac{1}{1 - \frac{r_g}{r}} + \delta y(r)$
solutions:	$\begin{cases} n = 2 & y(r) = \frac{r k_1}{3r_g^2 - 7r_g r + 4r^2} + \frac{r^2 k_2}{3(3r_g^2 - 7r_g r + 4r^2)} + \frac{r_g r^2 x(r) + 2(r_g r^3 - r^4) x'(r)}{(3r_g - 4r)(r_g - r)^2} \\ n \neq 2 & \text{whatever functions } x(r), y(r) \text{ and } R(r) \end{cases}$
$f(R)$ - theory:	$R/(\xi + R)$
first order metric:	$a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$
solutions:	$\begin{cases} x(r) = -\frac{4e^{-\frac{\xi^{1/2} r}{\sqrt{6}}}}{\xi} k_1 - \frac{2\sqrt{6}e^{-\frac{\xi^{1/2} r}{\sqrt{6}}}}{\xi^{3/2}} k_2 + k_3 r \\ y(r) = -\frac{2e^{-\frac{\xi^{1/2} r}{\sqrt{6}}}}{3b^{3/2}} (6\xi^{1/2} + \sqrt{6}\xi r) k_1 - \frac{2e^{-\frac{\xi^{1/2} r}{\sqrt{6}}}}{\xi^{3/2}} (\sqrt{6} - \xi^{1/2} r) k_2 \end{cases}$

TABLE II: A list of exact solutions obtained *via* the perturbation approach for several classes of  $f(R)$  theories;  $k_i$  are integration

## VII. CONCLUSIONS

General Relativity has been consistently tested in physical situations implying, essentially, spherical symmetry and weak field limit [49]. One of the fundamental and obvious issue that any theory of gravity should satisfy is the fact that, in absence of gravitational field or very far from a given distribution of sources, the spacetime has to be asymptotically flat (Minkowski). Any alternative or modified gravitational theory (beside the diffeomorphism invariance and the general covariance) should address these physical requirements to be consistently compared with GR. This is a crucial point which several times is not considered when people is constructing the weak field limit of alternative theories of gravity.

In this paper, we have faced this problem discussing, in the most general way, what the meaning of spherical solutions is in  $f(R)$  theories of gravity and when the standard results of GR are recovered in the limits  $r \rightarrow \infty$  and  $f(R) \rightarrow R$ . Essentially, spherical solutions can be classified, with respect to the Ricci curvature scalar  $R$ , as  $R = 0$ ,  $R = R_0 \neq 0$ , and  $R = R(r)$ , where  $R_0$  is a constant and  $R(r)$  is a function of the radial coordinate  $r$ . In these cases, the Birkhoff theorem holds; this means that stationary solutions are also static. However, as shown in the companion paper [34], this theorem does not hold, in general, for every  $f(R)$  theory since time-dependent evolution can emerge depending on the order of perturbations.

In order to achieve exact spherical solutions, a crucial role is played by the relations existing between the metric potentials and between them and the Ricci curvature scalar. In particular, the relations between the metric potentials and the Ricci scalar can be used as a constraint: this gives a Bernoulli equation. Solving it, in principle, spherically symmetric solutions can be obtained for any analytic  $f(R)$  function, both for constant curvature scalar and for curvature scalar depending on  $r$ .

Such spherically symmetric solutions can be used as background to test how generic  $f(R)$  theories of gravity deviate from GR. Particularly interesting are those theories that imply  $f(R) \rightarrow R$  in the weak field limit. In such cases, the experimental comparison is straightforward and also experimental results, evading GR constraints, can be framed in a self-consistent picture [26].

Finally, we have constructed a perturbation approach in which we search for spherically symmetric solutions at the 0th-order and then we search for solutions at the first order. The scheme is iterative and could be, in principle, extended to any order in perturbations. The crucial request is to take into account  $f(R)$  theories which are Taylor expandable about some value  $R = R_0$  of the curvature scalar.

A important remark is in order at this point. Considering interior and exterior solutions, the junction conditions are related to the integration constants of the problem and strictly depend on the source (e.g. the form of  $T$ ). We have not considered this aspect here since we have, essentially, searched for vacuum solutions. However, such a problem has to be carefully faced in order to deal with physically consistent solutions. For example, the Schwarzschild solution  $R = 0$ , which is one of the exterior solutions which we have considered, always satisfies the junction conditions with physically interesting interior metric. This is not the case for several spherically symmetric solutions which could give rise to unphysical junction conditions and not be in agreement with Newton's law of gravitation, also asymptotically. In these cases, such solutions have to be discarded. A detailed analysis in this sense is in the paper [42] for spherically symmetric solutions obtained in the Palatini formalism and in [43] for the metric approach.

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